M 317 Exam 1 all problems worth 30 pts

1. Define :

a) the **Least Upper Bound** of a nonempty set $A \subset R$.

$\alpha = Lub(A) \Leftrightarrow$ $i)$ $\alpha = UB(A)$ and $\alpha \leq \beta,\ \forall \beta = UB(A)$
$ii)$ $\alpha = UB(A)$ and $\forall \varepsilon > 0\ \ \alpha - \varepsilon \neq UB(A)$

b) an **accumulation point** of a nonempty set $A \subset R$

$p =$acc pt of A $\Leftrightarrow \forall \varepsilon > 0,\ \mathring{N}_\varepsilon(p) \cap A \neq \varnothing$

c) an **isolated point** of a nonempty set $A \subset R$

$p =$isol pt of A $\Leftrightarrow \exists \varepsilon > 0,\ \mathring{N}_\varepsilon(p) \cap A = \varnothing$

2. Prove **ONE** of the following:

a) If p is the least upper bound for a nonempty set $A \subset R,$ but p does not belong to $A,$ then p must be an accumulation point for $A.$

If $\alpha = Lub(A)$ then $\forall \varepsilon > 0,\ \alpha - \varepsilon \neq UB(A)$
Then $\exists x \in A$ such that $\alpha - \varepsilon < x \leq \alpha.$
But $\alpha \notin A,$ so $x \neq \alpha.$
That means $\alpha - \varepsilon < x < \alpha$ so $\forall \varepsilon > 0,\ \mathring{N}_\varepsilon(\alpha) \cap A \neq \varnothing$

b) If A is a nonempty open set in $R,$ then $\sup(A)$ does not belong to $A.$

If A is open then all pts in A are interior pts.
If $p = \sup(A) \in A$ then since $p \in A,\ \exists \varepsilon > 0,\ N_\varepsilon(p) \subset A,$ i.e.,$(p-\varepsilon, p+\varepsilon) \subset A.$
But if $p + \varepsilon \in A$ then $p \neq \sup(A),$ so $p = \sup(A)$ cannot belong to A

c) If p is be an accumulation point for A then A contains a sequence $\{a_n\}$ converging to $p.$

If $p =$acc pt of A then $\forall n,\ \mathring{N}_{1/n}(p) \cap A \neq \varnothing$
For each $n,$ choose $a_n \in \mathring{N}_{1/n}(p).$
Then $\forall n\ \ |a_n - p| < \frac{1}{n}\ if\ n > \frac{1}{\varepsilon}$ so $a_n \to p$

3. a) For the sequence $a_n = \dfrac{4n^3 + n^2 + 2}{n^3 + n^2 + 2},$ choose an L and find an N such that $|a_n - L| < 10^{-5}$ for $n > N.$

For large $n,\ a_n \approx 4$ so

$$|a_n - 4| = \left| \frac{4n^3+n^2+2}{n^3+n^2+2} - 4 \right|$$
$$= \left| \frac{4n^3+n^2+2}{n^3+n^2+2} - \frac{4(n^3+n^2+2)}{n^3+n^2+2} \right|$$
$$= \frac{|-3n^2-6|}{|n^3+n^2+2|} \leq \frac{3}{n} < 10^{-5} \quad \text{if } \; n > 3 \cdot 10^5$$

b) For $\lim_{x\to 2} \frac{x^2}{x^2+1}$, choose an L and a $\delta > 0$
such that $\left| \frac{x^2}{x^2+1} - L \right| < 10^{-5}$ for $|x-2| < \delta$

For $x \approx 2,\ \frac{x^2}{x^2+1}$ $\approx$ 4/5 so

$$\left| \frac{x^2}{x^2+1} - \frac{4}{5} \right| = \left| \frac{5x^2 - 4(x^2+1)}{5(x^2+1)} \right|$$
$$= \left| \frac{x^2-4}{5(x^2+1)} \right| < \frac{x+2}{5 \cdot 2}|x-2|$$

If $|x-2| < 1$, then $\frac{x+2}{10} \leq \frac{1}{10}$, so
$$\left| \frac{x^2}{x^2+1} - \frac{4}{5} \right| < \frac{1}{10}|x-2| < 10^{-5} \;\; if \;\; |x-2| < 10^{-4}$$

4. Is the sequence $a_n = \frac{(-1)^n n^2}{n^2+1}$ bounded?
$|a_n| = \frac{n^2}{n^2+1} < 1$, so it is a bounded sequence.

Does the sequence contain a convergent subsequence?
The B-W theorem asserts that every bounded sequence contains a cvg subsequence

Does the sequence $\{a_n\}$ converge?
Clearly a_{2n} converges to 1 while a_{2n+1} converges to -1, so by the uniqueness of limits theorem,
$\{a_n\}$ is not convergent.
State theorems which justify your answers.

5. Find the following function limits and state a theorem that proves the limit or one that asserts the limit fails to exist:

$a)\ \lim_{x\to\infty} \sin x$
$Let\ a_n = 2n\pi \;\; and \;\; b_n = 2n\pi + \pi/2.$ Then $\sin(a_n) = 0$ and $\sin(b_n) = 1,\ \forall n.$
Then by the uniqueness of limits theorem, the limit fails to exist.

$b)\ \lim_{x\to\infty} \frac{e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{e^{-2x}}{1 + e^{-2x}} = \frac{\lim_{x\to\infty} e^{-2x}}{\lim_{x\to\infty}(1 + e^{-2x})} = \frac{0}{1}$

by the arithmetic with limits theorem

alternatively, $0 < \frac{e^{-x}}{e^x + e^{-x}} < e^{-x}$ so the limit is 0 by the squeeze theorem

$c)\ \lim_{x\to 0} x^2 \cos x$

$-x^2 \le x^2 \cos x \le x^2$ so the limit is 0 by the squeeze theorem

alternatively,

$\lim_{x\to 0} x^2 \cos x = \lim_{x\to 0} x^2 \lim_{x\to 0} x^2 \cos x = 0 \cdot 1$ by the arithmetic with limits theorem